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A Nullstellensatz for ideals of C^∞ functions in dimension 2

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Abstract. Suppose that an ideal J of C^∞ functions on an open subset of $\mathbf{R}^2$ is a Łojasiewicz ideal. We describe the set of C^∞ functions vanishing on the zeros of J explicitly using J in an open neighborhood of each point in zeros of J, it can be obtained by taking real radical and closure starting from J repeatedly for a finite number of times. This gives an another affirmative answer to Bochnak's conjecture in dimension 2, which is first done by Risler.

Key words: Nullstellensatz, zero property, real radical, closed ideal, Łojasiewicz ideal.

1. Introduction

Let $U \subset \mathbf{R}^n$ be an open set and $\mathcal{E}(U)$ the ring of C^∞ functions on U. For an ideal $J \subset \mathcal{E}(U)$, let $Z(J)$ denote the zeros of J and $J^* \subset \mathcal{E}(U)$ the ideal of C^∞ functions vanishing on $Z(J)$. We say that J *has the zero property* if $J^* = J$ and that J *is real* if $g_1^2 + \cdots + g_k^2 \in J$ with all $g_i \in \mathcal{E}(U)$ implies all $g_i \in J$.

In 1973, Bochnak conjectured the following.

Bochnak's Conjecture ([4]) Suppose that $J \subset \mathcal{E}(U)$ is a finitely generated ideal. Then J has the zero property if and only if J is real and closed with respect to C^∞ topology.

In the same paper, Bochnak showed that if all of the generators are analytic, then J has the zero property if and only if J is real. As a corollary, he proved the following: Suppose that $f_1, \ldots, f_k \in \mathcal{E}(U)$ are analytic and all f_i have the zero property. Then the product function $f = f_1 \cdots f_k$ has the zero property if and only if the ideal (f) is real. In 1999, we gave a related result to the above Bochnak's result by getting rid of the analyticity condition on f_i and adding some topological conditions for zeros of f. In fact

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Theorem 1.1 ([6]) *Let M be a connected manifold of class C^∞ and k a positive integer. Suppose that $f_i \in C^\infty(M)$ have the zero property and that $f_i \not\equiv 0$ $(1 \le i \le k)$. Set $f = f_1 \cdots f_k$. Then the following seven conditions are equivalent.*

(1) *f has the zero property.*
(2) *(f) is real, i.e., ${g_1}^2 + \cdots + {g_p}^2 \in (f)$ implies $g_i \in (f)$ for $1 \le i \le p$.*
(3) *(f) is a radical, i.e., for some $k \in \mathbf{N}$, $g^k \in (f)$ implies $g \in (f)$.*
(4) *$\overline{G(f)} = V(f)$, where $V(f)$ denotes the zero set of f and $G(f)$ denotes the set of regular points of f in $V(f)$.*
(5) *$V(f_i) = \overline{V(f_i) \setminus V(f_j)}$ for $1 \le i, j \le k$, $i \ne j$.*
(6) *$V(f_i) = \overline{V(f_i) \setminus V(f_{j_1} \cdots f_{j_m})}$ for $1 \le m \le k-1$, $1 \le i, j_1, \ldots, j_m \le k$, $i \ne j_1, \ldots, j_m$.*
(7) *$V(f_i) = \overline{V(f_i) \setminus V(f_1 \cdots f_{i-1})}$ for $1 < i \le k$.*

In 1976, Risler [12] proved that Bochnak's conjecture is affirmative for $n = 2$, and in some restricted situation for $n = 3$. In the same year, Adkins and Leahy [3] showed that if an ideal $J \subset \mathcal{E}(U)$ is generated by analytic functions, then J^* coincides with the closure of the real radical of J with respect to C^∞ topology.

Recently, Acquistapace, Broglia and Nicoara [1] proved that if J is a Łojasiewicz ideal, then J^* coincides with the closure of the Łojasiewicz radical of J (see Definition 2.1). As applications they recovered the results of Bochnak for an ideal generated by analytic functions and of Adkins-Leahy for the closure of the real radical. Also, they defined the convexity of ideals and referred to the Bochnak's conjecture. They showed that a Łojasiewicz ideal J has the zero property if and only if it is closed, convex and radical. A convex radical ideal is a real ideal, but the converse is unknown, so Bochnak's conjecture is still an open problem.

According to the best of my knowledge, none gave J^* explicitly. However in the case of $\mathbf{R}^2$, we show that it is obtained by taking real radical and closure of J repeatedly.

Let $\mathcal{M}_p \subset \mathcal{E}(U)$ denote the ideal of C^∞ functions vanishing at $p \in U$ and $\mathcal{M}_p^k$ denotes its kth power. Let $J \subset \mathcal{E}(U)$ be an ideal. For $\psi \in J$ and $p \in U$, we define the order of ψ and J at p by $\mathrm{ord}_p\psi = \sup\{k | \psi \in \mathcal{M}_p^k\}$ and $\mathrm{ord}_p J = \inf\{\mathrm{ord}_p\psi \mid \psi \in J\}$. If $J = (f_1, \ldots, f_m)$ is finitely generated then we have $\mathrm{ord}_p f = 2\,\mathrm{ord}_p J$ for $f = f_1^2 + \cdots + f_m^2$.

We have the following.

Theorem 1.2 *Let $U \subset \mathbf{R}^2$ be open. Let $J \subset \mathcal{E}(U)$ be a Łojasiewicz ideal and $\{J_k\}$ defined by $J_0 = J$, $J_k = \overline{\sqrt[R]{J_{k-1}}}$ for $k \geq 1$. Let $p \in Z(J)$ and $s = \mathrm{ord}_p J$. Then there exists an open neighborhood U_p of p such that*

$$J^* \mathcal{E}(U_p) = \sqrt[R]{J_{2s-2}}\,\mathcal{E}(U_p).$$

Immediately we have the following, that is the affirmative answer to Bochnak's conjecture in dimension 2.

Corollary 1.3 (Risler) *Let $U \subset \mathbf{R}^2$ be open. If an ideal $J \subset \mathcal{E}(U)$ is finitely generated, real and closed with respect to C^∞ topology, then $J^* = J$.*

Proof. Since J is finitely generated and closed, J is a Łojasiewitz ideal from Proposition 2.2. Then Theorem 1.2 implies that, for any $p \in Z(J)$, there exists a neighborhood $U_p \subset U$ of p such that

$$J^* \mathcal{E}(U_p) = \sqrt[R]{J_{2s-2}}\ \mathcal{E}(U_p) = J\mathcal{E}(U_p) \quad (s = \mathrm{ord}_p J).$$

Therefore $J^* = \overline{J}$. Since J is closed, $J^* = J$. □

This paper is organized as follows. In Section 2, we mention some propositions needed later. In Section 3, we prove Theorem 1.2 by induction on $s = \mathrm{ord}_p J$.

2. Preliminaries

Definition 2.1 Let $J \subset \mathcal{E}(U)$ be an ideal. We say that J is a *Łojasiewicz ideal* if the following two conditions are satisfied.

(1) J is finitely generated.
(2) There exists an element $g \in J$ having the following property.
For any compact set $K \subset U$, there exist constants $C > 0$ and $\alpha \geq 0$ such that

$$|g(x)| \geq C d(x, Z(J))^\alpha \quad \text{for all } x \in K,$$

where $d(y, A)$ denote the Euclidean distance of y and A.

Proposition 2.2 *If J is finitely generated and closed, then J is a Łojasiewicz ideal.*

Proof. See [15, p. 103, Corollaire 4.4]. □

We say that $\psi \in \mathcal{E}(U)$ is *k-flat at p* if $\psi \in \mathcal{M}_p^{k+1}$ and that $\psi \in \mathcal{E}(U)$ is *flat at p* if $\psi \in \mathcal{M}_p^\infty = \cap_{k \in \mathbf{N}} \mathcal{M}_p^k$.

Proposition 2.3 *Let $J \subset \mathcal{E}(U)$ be a Łojasiewicz ideal. Then the following hold.*

(1) *If $\psi \in \mathcal{E}(U)$ is flat on $Z(J)$, then $\psi \in J$.*
(2) *There are no points $p \in U$ such that every $\psi \in J$ is flat at p.*

Proof. (1) See [15, p. 102, Proposition 4.3]. (2) If so, the inequality of the definition of Łojasiewicz ideal fails near p. □

Let $\sum^2 \subset \mathcal{E}(U)$ denote the set of sum of squares. We say that

$$\sqrt[R]{J} = \left\{\psi \in \mathcal{E}(U) \mid {}^\exists l \in \mathbf{N}, {}^\exists \sigma \in \textstyle\sum^2 \text{ s.t. } \psi^{2l} + \sigma \in J\right\}$$

is *a real radical* of J. This is an ideal and J is real if and only if $J = \sqrt[R]{J}$. Let $T_p : \mathcal{E}(U) \to \mathbf{R}[[x_1 - p_1, \ldots, x_n - p_n]]$ be the Taylor map at p.

Proposition 2.4 (Merrien's formal Nullstellensatz) *Let $J \subset \mathcal{E}(U)$ be a Łojasiewicz ideal. If $\psi \in J^*$, then for any $p \in Z(J)$ there exist a positive integer l and an element $\sigma \in \sum^2$ such that $\psi^{2l} + \sigma \in J + \mathcal{M}_p^\infty$.*

Proof. By applaying the formal Nullstellensatz ([12, Remarque 1.5]) for each $p \in Z(J)$, it holds that $T_p\psi \in \sqrt[R]{T_pJ}$ for all $p \in Z(J)$. Hence there exist a positive integer l and formal power series g_i for all $1 \leq i \leq k$ such that $(T_p\psi)^{2l} + g_1^2 + \cdots + g_k^2 \in T_pJ$. Then there exist elements $\sigma_i \in \mathcal{E}(U)$ such that $T_p\sigma_i = g_i$ for all $1 \leq i \leq k$. Putting $\sigma = \sigma_1^2 + \cdots + \sigma_k^2 \in \sum^2$, we have that $T_p\psi^{2l} + T_p\sigma \in T_pJ$. Therefore $\psi^{2l} + \sigma \in J + \mathcal{M}_p^\infty$. □

Proposition 2.5 (Whitney's spectral theorem [16]) *Let $J \subset \mathcal{E}(U)$ be an ideal and $\overline{J}$ the closure of J with respect to C^∞ topology. Then $\overline{J}$ coincides with the set of $\psi \in \mathcal{E}(U)$ such that $T_p\psi \in T_pJ$ for all $p \in Z(J)$.*

Proposition 2.6 *Let $U \subset \mathbf{R}^n$ be an open set and $J \subset \mathcal{E}(U)$ an ideal. If $V \subset U$ is open then*

(1) $\sqrt[R]{J\mathcal{E}(V)} = \sqrt[R]{J}\mathcal{E}(V)$,
(2) $\overline{J\mathcal{E}(V)} = \overline{J}\mathcal{E}(V)$.

Proof. See [2, Lemma 3.2 for (1), Lemma 1.1 for (2)]. □

3. Proof of Theorem 1.2

Let $f_1, \dots, f_m$ be generators of J, set $f = f_1^2 + \cdots + f_m^2$. From (2) of Proposition 2.3, $T_p f \neq 0$. Therefore there exist an open neighborhood of p and a local coordinates (t, x) centered at p such that $f(t, x)$ is regular of order $2s$ with respect to t, namely we may write $f(t, 0) = t^{2s}(c + g(t))$ with $c > 0$ and $g(0) = 0$. From the Malgrange preparation theorem, we may write

$$f(t,x) = \{t^{2s} + u_1(x)t^{2s-1} + \cdots + u_{2s}(x)\}Q(t,x), \quad Q(0,0) > 0$$

in an open neighborhood U_0 of the origin of $\mathbf{R}^2$. Moreover we may assume that $u_1(x) \equiv 0$ and U_0 is convex by a suitable coordinate change.

Then we define $F \in \mathcal{E}(U_0)$ by

$$F(t,x) = t^{2s} + u_2(x)t^{2s-2} + \cdots + u_{2s}(x).$$

To prove that $J^*\mathcal{E}(U_0) = \sqrt[R]{J_{2s-2}}\mathcal{E}(U_0)$, it is sufficient to show that $\varphi|_{U_0} \in \sqrt[R]{(F)_{2s-2}}$ for all $\varphi \in J^*$ from Proposition 2.6.

We will prove by induction on s.

Step 1. In the case of $s = 1$.

In this situation, $F = t^2 + u_2$. Notice that the principle ideal (u_2) is a Łojasiewicz ideal as well as F. Let $\psi = \varphi|_{U_0}$ for $\varphi \in J^*$.

(1) In the case of $u_2 \equiv 0$.

It follows that $\psi(0, x) = 0$ since $Z(F) = \{(t, x) \in U_0 | t = 0\}$. Then there exists an element $\eta \in \mathcal{E}(U_0)$ such that $\psi = t\eta$ since U_0 is convex. Obviously $t \in \sqrt[R]{(F)}$, hence we obtain that $\psi \in \sqrt[R]{(F)}$.

(2) In the case of $u_2 \not\equiv 0$.

From (2) of Proposition 2.3, it holds that $T_0 u_2 \neq 0$. Since u_2 is one variable, we have that $Z(u_2) = \{0\}$ in an open neighborhood of $0 \in \mathbf{R}$. Therefore $Z(F) = \{(0,0)\}$ in an open neighborhood of $(0,0) \in \mathbf{R}^2$. Then we replace U_0 with this open neighborhood. From Lemma 2.4, there exist a positive integer l and elements $\sigma \in \sum^2$, $\eta \in \mathcal{E}(U_0)$, $\theta \in \mathcal{M}^\infty_{(0,0)}$ such that $\psi^{2l} + \sigma = F\eta + \theta$. From (1) of Proposition 2.3, it follows that $\theta \in (F)$. Hence it holds that $\psi^{2l} + \sigma \in (F)$. Therefore we obtain that

$\psi \in \sqrt[R]{(F)}$.

Step 2. Suppose that $s \geq 2$ and Theorem 1.2 is true for smaller s.

We observe $F \in \mathcal{E}(U_0)$ such that $F = t^{2s} + u_2(x)t^{2s-2} + \cdots + u_{2s}(x)$. Let $Z_r = \{q \in Z(F) | \mathrm{ord}_q J = r\}$. Then it follows that $Z(F) = (Z_1 \cup \cdots \cup Z_{s-1}) \cup Z_s$. Let us put $\psi = \varphi|_{U_0}$ for $\varphi \in J^*$. From the hypothesis of induction, for any $q \in Z_r$ $(1 \leq r \leq s-1)$, there exists an open neighborhood $U_q \subset U_0$ of q such that $J^*\mathcal{E}(U_q) = \sqrt[R]{J_{2r-2}}\mathcal{E}(U_q)$. Clearly it holds that $J_{2r-2} \subset J_{2s-4}$ for all $1 \leq r \leq s-1$. Hence it follows that

$$(*) \qquad T_q\psi \in T_q \sqrt[R]{(F)_{2s-4}} \text{ for all } q \in Z(F) \setminus Z_s.$$

Next, we will observe neighborhoods of points in Z_s. Set $\delta = u_2^2 + \cdots + u_{2s}^2$. Then $(t, x) \in Z_s$ implies $t = 0$ and $\delta(x) = 0$.

Let $(0, x_0) \in Z_s$ be arbitrarily fixed.

(1) In the case of $T_{x_0}\delta \neq 0$.

There exists an open neighborhood V_{x_0} of x_0 such that $Z(\delta) \cap V_{x_0} = \{x_0\}$ since δ is one-variable. Hence there exists an open neighborhood $W_{(0,x_0)}$ of $(0, x_0)$ such that $Z_s \cap W_{(0,x_0)} = \{(0, x_0)\}$. Then it follows that

$$T_{(0,x_0)}(\psi^{2l} + \sigma) \in T_{(0,x_0)}(F) \quad (\sigma \in \Sigma^2)$$

from Proposition 2.4 and

$$T_{(t,x)}\psi \in T_{(t,x)} \sqrt[R]{(F)_{2s-4}} \text{ for all } (t, x) \in W_{(0,x_0)} \setminus (0, x_0)$$

from $(*)$. Hence it holds that $T_{(t,x)}\psi(\psi^{2l} + \sigma) \in T_{(t,x)} \sqrt[R]{(F)_{2s-4}}$ for all $(t, x) \in W_{(0,x_0)}$. Therefore from Proposition 2.5 and (2) of Proposition 2.6, it follows that

$$\begin{aligned}
\psi^2(\psi^{2l} + \sigma) \,|_{W_{(0,x_0)}} &\in \overline{\sqrt[R]{(F)_{2s-4}}\mathcal{E}(W_{(0,x_0)})} \\
&= \overline{\sqrt[R]{(F)_{2s-4}}}\mathcal{E}(W_{(0,x_0)}) \\
&= (F)_{2s-3}\mathcal{E}(W_{(0,x_0)}).
\end{aligned}$$

Hence from (1) of Proposition 2.6, it holds that

$$\begin{aligned}\psi|_{W_{(0,x_0)}} &\in \sqrt[R]{(F)_{2s-3}\mathcal{E}(W_{(0,x_0)})}\\ &= \sqrt[R]{(F)_{2s-3}}\mathcal{E}(W_{(0,x_0)}).\end{aligned}$$

Therefore we obtain that

$$T_{(0,x_0)}\psi \in T_{(0,x_0)} \sqrt[R]{(F)_{2s-3}}.$$

(2) In the case of $T_{x_0}\delta = 0$.

From Proposition 2.4, there exist a positive integer l and an element $\sigma \in \sum^2$ such that $\psi^{2l} + \sigma \in (F) + \mathcal{M}^\infty_{(0,x_0)}$. Since $T_{x_0}\delta = 0$, it holds that $\psi^{2l} + \sigma \in (t) + \mathcal{M}^\infty_{(0,x_0)}$. Hence we have that $\psi(0,x)^{2l} + \sigma(0,x) \in \mathcal{M}^\infty_{x_0}$. Therefore it holds that $\psi(0,x) \in \mathcal{M}^\infty_{x_0}$. Hence it follows that $\psi \in (t) + \mathcal{M}^\infty_{(0,x_0)}$. Then it holds that

$$\psi^{2s} \in (t^{2s}) + \mathcal{M}^\infty_{(0,x_0)} = (F) + \mathcal{M}^\infty_{(0,x_0)}.$$

Hence it follows that

$$T_{(0,x_0)}\psi^{2s} \in T_{(0,x_0)}(F).$$

From (1) and (2), it follows that

$$(**) \qquad T_{(0,x_0)}\psi^{2s} \in T_{(0,x_0)} \sqrt[R]{(F)_{2s-3}} \text{ for all } (0,x_0) \in Z_s.$$

The inclusions $(*)$ and $(**)$ imply that $T_{(t,x)}\psi^{2s} \in T_{(t,x)} \sqrt[R]{(F)_{2s-3}}$ for all $(t,x) \in Z(F)$. From Proposition 2.5, it follows that

$$\psi^{2s} \in \overline{\sqrt[R]{(F)_{2s-3}}} = (F)_{2s-2}.$$

Thus we obtain $\psi \in \sqrt[R]{(F)_{2s-2}}$. □

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